Matrix Multipliers

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The elementary multiplier concept claims only to explain how an increase in final demand, such as a step up in the level of government spending on goods and services, will affect the level of aggregate Gross Domestic Product. The multiplier neglects the obvious limitations involved in working with such national income aggregates as GDP and government spending without regard to the composition of government spending and output. For example, the multiplier will not tell us what will happen if the government shifts $50 billion from military spending to the construction of urban transportation networks or to public housing. Such a shift would obviously stimulate the housing industry. It might be bad for electronics. But would it cause an increase or a decrease in the demand for steel? The technique of "input-output" analysis, developed by Nobel Laureate Wassily Leontief in the 1930's, may be used to analyze the effects of shifts in the composition of government spending on the output of different industries. This procedure for addressing inter-industry complications is easily understood with the aid of matrix algebra.

1: The Input-Output Model:

In its simplest form input-output analysis supposes that there is a large number, say n, commodities (e.g., eggs, cookies, steel, pig iron, automobiles, or frozen pork bellies), each produced by a specific technology (recipe) with two specific characteristics, fixed proportions and constant returns to scale.

> An example of a commodity would be cookies: The technological assumptions imply that there is only one recipe (fixed proportions) for producing cookies; it is also assumed that if the quantity of each input called for by the recipe is doubled, output will double as well (constant returns to scale).

In order to work out the implications of these technological assumptions, let X_i denote the output of commodity i, which is used in part to meet the intermediate requirements as input in the production of other commodities and in part to meet final demand (e.g. government purchases, investment and consumption). If X_{ij} denotes the quantity of i used to produce a unit of good j and Y_i denotes the final demand for good i we have the "materials balance" condition:

$$X_i = X_{i1} + X_{i2} + ... + X_{in} + Y_i\,, \quad (i = 1,...n) \qquad (1)$$

That is to say, the output of each commodity just suffices to meet the sum of interindustry plus final demand for that commodity.[1] There must be n such equations, one for each of the n commodities produced in the economy.

Now let a_{ij} denote the quantity of the commodity i that is used to produce a unit of commodity j. Then we will have

$$X_{ij} = a_{ij}X_j \qquad (i = 1,...n; j = 1,...n) \qquad (2)$$

[1] It is assumed in static input-output analysis that final demand, including consumption demand and changes in the inventory component of investment spending, is exogenous. Only a slight algebraic modification of the basic model is required to make consumption endogenous, as in the aggregate multiplier model. Dynamic input-output models make investment endogenous.

For example, if our cookie recipe calls for three eggs to produce a dozen cookies, we would have $a_{e,c}$ = 3, where the subscripts e and c refer to eggs and cookies respectively. Then if we are to produce 5 dozen cookies we will require $a_{e,c} X_c = 3x5 = 15$ eggs.

Substituting from (2) into (1) we obtain a system of n simultaneous non-homogeneous equations:

$$X_i = a_{i1}X_1 + a_{i2}X_2 + ... + a_{in}X_n + Y_i, \quad (i = 1,...n). \qquad (3)$$

The task of input-output analysis is to solve this system of equations for the n gross industry outputs X_i, given the input-output coefficients a_{ij} and the final industry demands Y_i. But first it is necessary to estimate the a_{ij}.

2: Empirical Implementation

Leontief used U.S. Census of Manufacturing data to estimate the parameters of his model. The census provided raw data on interindustry flows (X_{ij}) and final demand (Y_i). From this the a_{ij} input-output coefficients are calculated: $aij = Xj/X_{ij}$, as suggested by equation (2). Such systems have been estimated for many countries, some involving several hundred industries. Matrices of input-output coefficients have been published on a regular basis by the Department of Commerce for several decades.

3: Matrix Notation

In matrix notation this system of equations (2) may be compactly expressed as

$$X = AX + Y, \qquad (4)$$

where $A = [a_{i,j}]$ is the nxn matrix of input-output coefficients, $X = col[X_j]$ is the an n component column vector of gross outputs, and $Y = col[Y_j]$ is the vector of final demand. The components of the vector X are of particular interest: the x_i, are the total output requirements of commodity i, including the sum of the direct requirements of i in the production of Y_i plus the output of i used indirectly for the production of goods that are used as inputs in the production of j, etc. The matrix $A = [a_{ij}]$ is sometimes called the matrix of "direct requirements" in that it shows the quantity of good i required directly in the production of 1 unit of commodity j.

To solve this system for the vector of gross outputs X as a function of the final demand vector Y, we first subtract AX from both sides of (4):

$$X - AX = [I-A]X = Y \qquad (5)$$

Provided the matrix [I-A] is non-singular, pre-multiplication of (5) by $[I-A]^{-1}$ yields the desired vector of gross outputs as a function of the vector of final demand:

$$X = [I-A]^{-1}Y \qquad (6)$$

The total requirements of good i in the production of a unit of good j, both direct plus indirect, is revealed by the i,jth element of the Leontief inverse matrix $[I-A]^{-1.}$

4: Exercise

The following table reports Census of Manufacturing data for a hypothetical economy which produces only two commodities, iron and steel:

	Coal	Steel	Final Demand	Gross Output
Coal	1,800	250	200	2,250
Steel	225	875	150	1,250

The Coal row reveals that 1800 units of coal were used to produce coal, 250 tons to produce steel, and that final demand was 200; thus the total output of 2,250 tons of coal just sufficed to meet these requirements. Obviously, these data are not meant to be realistic, but they will suffice to illustrate the calculations involved in input-output analysis.

1. Calculate the 2x2 matrix A showing the direct requirements of good i in the production of good j. [Hint: a_{12} = 250/1250 = 0.2].

2. Calculate the 2x2 matrix $[I-A]^{-1}$ showing the total requirements (direct plus indirect) of producing each commodity. [Hint: The total requirement of steel per unit of coal is = 2.5].

3. Suppose that the vector of final demand changes to Y = col(160, 180). Calculate the new vector of gross output $X = [I-A]^{-1}Y$, assuming that the input-output coefficients $a_{i.j}$ do not change.

5. Other Applications:

5.1. International Trade:

While the simplest version of the aggregate multiplier neglects complications created by international trade, it is easily modified to allow for the leakage of effective demand to other countries as a result of the increase in imports that will be generated when output expands as a result of an increase in government spending — the resulting “foreign trade multiplier” is somewhat smaller than that calculated without regard to trade complications. But this minor refinement may understate the effect on domestic output of the increase in government spending because the rise in American imports may also stimulate the economies of our trading partners, inducing them to purchase more goods from the United States.

Such multi-country feedback complications may be readily taken into account by redefining the variables used in input-output analysis: Let $X_{i,j}$ now denote the exports of country i to country j ($X_{i,i}$ is domestic utilization by country i of its own output), Y_i denotes government spending plus investment in country i, and X_i denotes the GDP of country i. Then $a_{ii} = X_{ii}/X_i$ is the marginal propensity of country i to consume its own output and $a_{ij} = X_{ij}/X_j$, $i \neq j$, is the marginal propensity of country j to import goods from country i. The diagonal elements of the Leontief inverse matrix $[I-A]^{-1}$ now reveal the “own multiplier” effects of an increase in a country's government spending on its GDP while the off-diagonal coefficients show how the effects of an increase in government spending in any one country are distributed internationally.

5.2. Regional Economics:

An increase in government spending in one region, say Massachusetts, will stimulate that region's economy, but since many of the materials used for the project are likely to be produced outside of that state, the multiplier effects will be dissipated. That is why an effort by a single city or even a state to stimulate its economy through tax cuts or an increase in government spending is likely to spill over into neighboring regions while having a rather small effect on the local economy.

The foreign trade multiplier concept can easily be applied to analyze this complication by adopting the convenient fiction of treating each region as a separate country. More than this, one can construct a regional input-output model by treating steel produced in South Bend, Indiana, as being a different commodity from steel produced in Pittsburgh. Such models have been used in analyzing the effects of military expenditures and disarmament on different regions of the country.

6. Warning

One basic limitation of Leontief's input-output approach is that it neglects the possibility of substitution of one input for another in the production process. It assumes that there is a single unique production process (i.e., only one recipe) for producing each good. It thus neglects the important point that a shift in the composition of final demand that leads to shortages in certain sectors of the economy is likely to lead to increases in the costs of certain inputs and a resultant shift in the input mix used by production processes. In defense, it can be asserted that it is not all that difficult to set up the analysis in terms of a group of profit maximizing firms driven to zero profits as a result of competition. The hard thing is to empirically implement such a model. Thus it is the empirical intractability of more involved production functions that may justify this simplified mode of analysis.

A second objection is that input-output analysis generates poor predictions. It is said that a cheaper but no less effective alternative to estimating the effect of a shift in the composition of government demand with input-output analysis is to invoke the naive assumption that each sector's output will expand in proportion to the change in the final demand for that commodity's output. In rebuttal, some defenders of input-output analysis assert that this may be true for relatively small changes but in times of wartime mobilization when there may be stupendous changes in the composition of government demand the input-output mode of analysis may do much better than the naive model.

7. Viable Solutions: A Theorem

It would obviously be disastrous to invest the great amount of effort involved in estimating an empirical input-output table only to find that the system yielded no predictions because the matrix [I-A] turned out to be singular; it would be equally unfortunate if the system ended up predicting negative "outputs" for certain sectors (e.g. steel output being negative, the blast furnaces operating in reverse to produce coal and coke from steel — disposal of scrap is not that simple. Fortunately, it can be shown that these incongruities can not arise when the matrix A is estimated from Census data.

First we define a convenient concept:

DEFINITION: A non-negative nxn matrix A is said to be *productive* if there exist n-component column vectors $X > 0$ and $Y > 0$ such that $X - AX = (I-A)X = Y$.

This condition says that for each of the n sectors gross outputs must exceed the intermediate use of its output in the production of other commodities. Clearly, census year data collected for an empirical input-output study will yield an input-output matrix A and vectors of gross outputs and final demand satisfying this condition.

Next it is useful to prove the following:

LEMMA: If the nxn matrix $A \geq 0$ is productive then $\lim_{t\to\infty} A^t = 0$.

- Proof: Let $\alpha_i = (x_i - y_i)/x_i$; since A is productive, $0 \leq \alpha_i < 1$.
  - Now if we let $\alpha = \max(\alpha_i)$, then $AX < \alpha X$ with $0 \leq \alpha < 1$.
  - If for some positive integer t, $A^t X < \alpha^t X$, then
  - $A^{t+1}X < A(\alpha^t X) = \lambda^t(\alpha X) < \alpha^{t+1}X$.
  - So by induction it follows that for any positive integer t,
  - $A^t X < \alpha^t X$.
  - Also, $A \geq 0$ and $X > 0$ implies $A^t X \geq 0$.
  - Therefore, since $\operatorname{limit}_{t\to\infty} \alpha^t = 0$, $\operatorname{limit}_{t\to\infty} A^t = 0$, as was to be shown.

Now we shall prove a remarkable proposition:

THEOREM: If the nxn matrix A is productive then $[I-A]^{-1} > 0$ and

$$[I-A]^{-1} = I + A + A^2 + \ldots \tag{7}$$

PROOF: Let $S_t = I + A + A^2 + \ldots + A^t$. Then $AS_t = A + A^2 + \ldots + A^{t+1}$, and $[I-A]S_t = I - A^{t1}$. It follows from the Lemma that $[I-A] \operatorname{limit}_{t\to\infty} S_t = I$. Hence the inverse of [I-A] is the non-negative matrix

$$S = \operatorname{limit}_{t\to\infty} S_n = I + A + A^2 \ldots$$

OBSERVATION: The iterative procedure suggested by (7) constitutes a simple but not necessarily efficient technique for inverting large input-output matrices.

NOTE: Students familiar with the concept of eigenvalues (characteristic roots) may note that for any square matrix A, $\operatorname{limit}_{t\to\infty} A^t = 0$ only if all the eigenvalues of A are less than one in absolute value. Many linear algebra texts discuss the Leontief model or the closely related von Neumann model of the expanding economy. For example, see Gilbert and Strong, *Linear Algebra*, page 204.

References:

Chipman, John S., *The Theory of Inter-Sectoral Money Flows and Income Formation*, 1951: Johns Hopkins University Press.

Debreu, Gerard and I. N. Herstein, “Nonnegative Square Matrices,” *Econometrica*, 21 (October 1953), pp 597-607.

Leontief, Wassily, *The Structure of the American Economy*, 1919-1939.

Leontief, Wassily, Alison Morgan, Karen Polenske, David Simpson and Edward Tower, “The Economic Impact — Industrial and Regional — of an Arms Cut,” *Review of Economics and Statistics*, August 1965, pp 217-241.