Two Dynamic Models – Cycles and Growth

1. Multiplier-Accelerator Interaction

We will consider what may well be the simplest macroeconomic model capable of generating a business cycle. This model shows that it is not necessary to advance separate explanations of the upper and lower turning points in order to account for cyclical phenomena. And it is not necessary to invoke a devil's teory explaining the occurrences of economic downturns as being the consequence of unfortunate errors made by either the monetary or the fiscal authorities, of wars or OPEC price hikes or productivity shocks,. The model, developed near the end of the Great Depression by Alvin Hansen and Paul Samuelson,[1] is outdated but simple It neglects depreciation, foreign trade, supply side shocks, monetary policy and a host of other complications that can be considered with more elaborate models of cyclical fluctuations.

1.1. Assumptions:

Consumption is determined by last period's income:

$$C_t = c_0 + c_1C_{t-1}. \tag{1}$$

The capital stock (machinery, equipment and factories) that entrepreneurs desire to process is proportional to output.

$$K_t^d = kY_{t-1}. \tag{2}$$

Net Investment, by definition, is

$$I_t = K_t – K_{t-1}. \tag{3}$$

Firms always undertake enough net investment to keep their capital stock at the desired level:

$$I_t = K_t^d – K_{t-1}. \tag{4}$$

Therefore, $K_t^d = K_t = kY_{t-1}$ and $K_{t-1}^d = K_{t-1} = kY_{t-2}$, so we have the "accelerator" explanation of investment:

$$I_t = k(Y_{t-1} –Y_{t-2}). \tag{5}$$

Finally, since $Y_t = C_t + I_t + G_t$, substitution yields a second order linear difference equation explaining the of our economies law of motion:

$$Y_t = c_0 + (c_1+k_1)Y_{t-1} – kY_{t-2} + G_t. \tag{6}$$

1.2. Dynamics

The first simulation on Figure 1.1 illustrates the type of behavior generated by this system for $c_0 = 0$, $c_1= 0.5$, and $k = 1$. Initially, the system is in equilibrium for G = 50 with Y =100, C + 50 and I = 0. But in period 2 government spending steps up to G = 100 and remains at this higher level for ever more. The graph reveals the persistent cycle that results! The cycle is damped and Y approaches 200 in the limit. The limiting value of GDP is correctly predicted by the multiplier.

The second simulation shows that the model can generated convergent behavior. For the chozen parameter values GDP approaches 200 in the limit. The third simulations show that the model can also generate divergent oscillations. Monotonic convergence to equilibrium is also possible, as with $c_1 = 0.9$ and $k = 0.5$.

The model's parameters determine the type of motion. It can be shown, by analyzing the roots of the characteristic equation of (6), that stability requires $k < 1$. The limiting value of GDP approached by a stable system is correctly predicted by the multiplier

[1] Paul A. Samuelson, "Interactions Between the Multiplier Analysis and the Principle of Acceleration," *Review of Economic Statistics,* May, 1939.

2. Solow's Growth Model

What determines in the long-run whether an economy will grow or decay? What determines the rate of growth? And why do some countries remain dormant while others takeoff into self-sustained growth. A pioneering contribution toward the resolution of such questions was provided by the Solow model of economic growth.[2]

2.1. Assumptions:

Output Q is a function of labor L, capital K, and land R. Since all but R evolve over time, we could write Q(t), L(t), and K(t), but we suppress the time notation. The level of output is determined by the following production function

$$Q = \alpha(1+\rho)^t L^{\lambda} K^{\lambda'} R^{1-\lambda-\lambda'}, \tag{7}$$

This elaborates on the Cobb-Douglas production function (equation (6) of Chapter 5) in two respects: First, it includes R for resources in fixed supply, e.g., land, as an additional input. With $\lambda + \lambda' < 1$, the function is not homogeneous of degree one in capital and labor; we have diminishing returns to scale in the two variable inputs, implying that a doubling of labor and capital would not double output. Second, technological progress is captured by the term $(1+\rho)^t$. With $\rho > 0$, this means that if L, K, and R were to remain unchanged output would still grow with the passage of time because of improved techniques of production.

It is also assumed that the population grows at constant rate n:

$$N = N_0(1+n)^t. \tag{8}$$

Further, a constant portion γ of the population is employed. Presumably, the labor force participation rate is constant and the employed proportion of the labor force does not vary, either because of the economy's natural self-recuperating powers or because the central bankers succeeds in keeping the economy moving along its full-employment growth path. Therefore, the labor supply grows at rate n.

$$L = \gamma N = \gamma N_0(1+n)^t. \tag{9}$$

In addition, suppose that a constant fraction s of output is saved. Then consumption is C = (1-s)Q. Since there is no government or foreign trade, Q = C + I and we have net investment

$$I = dK/dt = sQ \tag{10}$$

2.2. Analysis

Notation: Lower case letters denote rates of growth. For example, $k = \dfrac{dK/dt}{K}$.

As a first step toward determining the laws of motion of this dynamic model, we ask whether output can grow at a constant rate, call it q^e. To find out, let us first take logs to the base e of (7), with the approximation $\ln(1+\rho) = \rho$:

$$\ln Q = \ln\alpha + (1-\lambda - \lambda')\ln R + t\rho + \lambda \ln N + \lambda' \ln K. \tag{11}$$

Differentiating with respect to t yields

$$\frac{dQ/dt}{Q} \equiv q = \rho + \lambda n + \lambda' k \tag{12}$$

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[2] Robert M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics,* February, 1956.

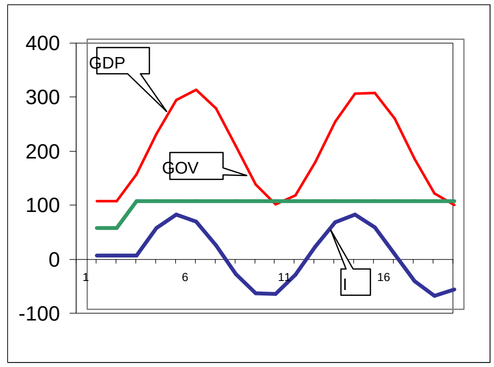
Figure 1.1

Multiplier-Accelerator Interaction

Simulation I

Parameters: $c_0 = 0$ $c_1 = 0.5$ $k = 1$

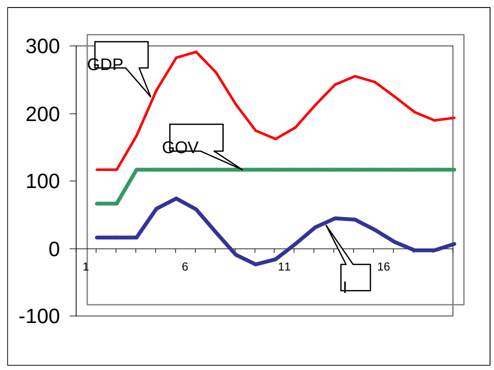
date	GDP	Consump	Invest	GovSpend	KapStock	DesiredK
	$Y=C+I+G$	$C=c_0+c_1Y_{t-1}$	$I=K_t^d-K_{t-1}$	G_t	$K_t=K_{t-1}+I_t$	$K_t^d=kY_{t-1}$
0	100.0	50.0	0.0	50.0	100.0	
1	100.0	50.0	0.0	50.0	100.0	100.0
2	150.0	50.0	0.0	100.0	100.0	100.0
3	225.0	75.0	50.0	100.0	150.0	150.0
4	287.5	112.5	75.0	100.0	225.0	225.0
5	306.3	143.8	62.5	100.0	287.5	287.5
6	271.9	153.1	18.8	100.0	306.3	306.3
7	201.6	135.9	-34.4	100.0	271.9	271.9
8	130.5	100.8	-70.3	100.0	201.6	201.6
9	94.1	65.2	-71.1	100.0	130.5	130.5
10	110.7	47.1	-36.3	100.0	94.1	94.1
11	172.0	55.4	16.6	100.0	110.7	110.7
12	247.2	86.0	61.2	100.0	172.0	172.0
13	298.9	123.6	75.2	100.0	247.2	247.2
14	301.1	149.4	51.6	100.0	298.9	298.9
15	252.7	150.5	2.2	100.0	301.1	301.1
16	178.0	126.4	-48.3	100.0	252.7	252.7
17	114.3	89.0	-74.7	100.0	178.0	178.0
18	93.5	57.2	-63.7	100.0	114.3	114.3



Simulation II

Parameters: $c_0 = 0$ $c_1 = 0.5$ $k = 0.85$

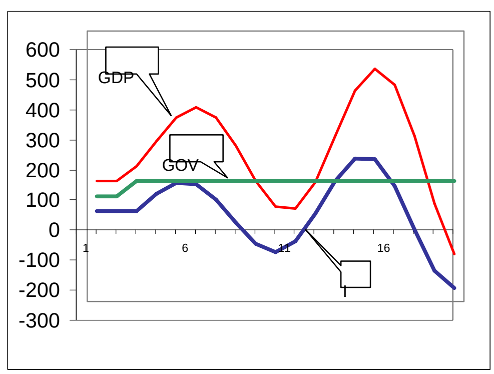
date	GDP	Consump	Invest	GovSpend	KapStock	DesiredK
	$Y=C+I+G$	$C=c_0+c_1Y_{t-1}$	$I=K_t^d-K_{t-1}$	G_t	$K_t=K_{t-1}+I_t$	$K_t^d=kY_{t-1}$
0	100.0	50.0	0.0	50.0	85.0	
1	100.0	50.0	0.0	50.0	85.0	85.0
2	150.0	50.0	0.0	100.0	85.0	85.0
3	217.5	75.0	42.5	100.0	127.5	127.5
4	266.1	108.8	57.4	100.0	184.9	184.9
5	274.4	133.1	41.3	100.0	226.2	226.2
6	244.2	137.2	7.0	100.0	233.2	233.2
7	196.5	122.1	-25.6	100.0	207.6	207.6
8	157.6	98.2	-40.6	100.0	167.0	167.0
9	145.8	78.8	-33.0	100.0	134.0	134.0
10	162.9	72.9	-10.1	100.0	123.9	123.9
11	195.9	81.4	14.5	100.0	138.4	138.4
12	226.1	98.0	28.1	100.0	166.5	166.5
13	238.6	113.0	25.6	100.0	192.1	192.1
14	230.0	119.3	10.7	100.0	202.8	202.8
15	207.7	115.0	-7.3	100.0	195.5	195.5
16	184.9	103.8	-19.0	100.0	176.5	176.5
17	173.0	92.4	-19.4	100.0	157.1	157.1
18	176.5	86.5	-10.1	100.0	147.1	147.1



Simulation III

Parameters: $c_0 = 0$ $c_1 = 0.5$ $k = 1.15$

date	GDP	Consump	Invest	GovSpend	KapStock	DesiredK
	$Y=C+I+G$	$C=c_0+c_1Y_{t-1}$	$I=K_t^d-K_{t-1}$	G_t	$K_t=K_{t-1}+I_t$	$K_t^d=kY_{t-1}$
0	100.0	50.0	0.0	50.0	115.0	
1	100.0	50.0	0.0	50.0	115.0	115.0
2	150.0	50.0	0.0	100.0	115.0	115.0
3	232.5	75.0	57.5	100.0	172.5	172.5
4	311.1	116.3	94.9	100.0	267.4	267.4
5	346.0	155.6	90.4	100.0	357.8	357.8
6	313.1	173.0	40.1	100.0	397.9	397.9
7	218.7	156.5	-37.8	100.0	360.0	360.0
8	100.8	109.3	-108.5	100.0	251.5	251.5
9	14.8	50.4	-135.6	100.0	115.9	115.9
10	8.6	7.4	-98.9	100.0	17.1	17.1
11	97.0	4.3	-7.2	100.0	9.8	9.8
12	250.3	48.5	101.8	100.0	111.6	111.6
13	401.4	125.1	176.2	100.0	287.8	287.8
14	474.4	200.7	173.8	100.0	461.6	461.6
15	421.2	237.2	84.0	100.0	545.6	545.6
16	249.4	210.6	-61.2	100.0	484.4	484.4
17	27.2	124.7	-197.6	100.0	286.9	286.9
18	-142.1	13.6	-255.6	100.0	31.2	31.2



Since n is the constant rate of growth of the labor force, this equation says that if output grows at a constant rate q then k, the rate of growth of the capital stock, must also be constant. More than this, from (10) we have

$$sQ/K = I/K = \frac{dK/dt}{K} = k \tag{13}$$

This means that the capital stock can grow at a constant rate k only if the output capital ratio, Q/K is constant, but that requires that Q and K grow at the same rate; i.e. $k = q^e$ if output grows at a constant rate. To find q^e, substitute it for q and k in (12) to obtain:

$$q^e = \rho + \lambda n + \lambda' q^e = \frac{\rho + \lambda n}{1 - \lambda'} \tag{14}$$

The rate of growth of output per capita is $q^e - n$. Per capita income will increase along the equilibrium growth path, output growing faster than the population, if and only if

$$q^e - n = \frac{\rho + \lambda n}{1 - \lambda'} - n > 0, \text{ or } \rho > (1 - \lambda - \lambda')n. \tag{15}$$

The properties of this equilibrium are clarified with the aid of Figure 2.1, which plots the output/capital ratio on the abscissa and rates of growth on the ordinate. The ray emanating from the origin denotes the equation k = sQ/K, from (13). The line labeled q is obtained by substituting k = sQ/K into (12) to obtain

$$q = \rho + \lambda n + \lambda' sQ/K. \tag{16}$$

The slope of the k line is sQ/K, which means that it is steeper than the q line because its slope is only λ'sQ/K. Hence the two lines must intercept. The interception point, marked e on the graph , is a point of equilibrium because if the output capital ratio economy ever moves marks the intersection point. , The point where the two lines cross, where q = k, yields simultaneously the equilibrium values q^e and $(Q/K)^e$. Since $q^e = k^e = I/K$, equation (16) yields

$$\left(\frac{Q}{K}\right)^e = \frac{q^e}{s} = \frac{\rho + \lambda n}{s(1 - \lambda')}. \tag{17}$$

This growth equilibrium is stable. To see why, suppose that initially the output/capital ratio is $\left(\frac{Q}{K}\right)_0 > \left(\frac{Q}{K}\right)^e$, as illustrated on Figure 2.2. This would be the situation for a country that has yet to realize its development potential. Since its output/capital ratio is high, $q > q^e$, as can be seen from equation (16); i.e., our country will be growing above its equilibrium rate. But k > q, implying that the output capital ratio will fall with the passage of time. Thus Q/K will approach its equilibrium value as a limit, as indicated by the arrows on Figure 2.2.

The first simulation on Figure 2.3 shows the process by which the country may gradually move toward happy equilibrium. Note that the adjustment process can be quite slow, but the end result is a country cruising along its full-employment growth path with a stable capital/output ratio and a constant rate of growth of output and the capital stock. Simulation II shows, for a different set of parameters, a not so happy case in which the rate of growth of output is less than the rate of population growth, which means that the standard of living must inevitably decline.

It is intriguing to note from equation (14) that the equilibrium rate of growth does *not* depend on s, the proportion of output that is saved for reinvestment rather than consumed. Compare Simulation III with Simulation I. However, s does affect the equilibrium capital/output ratio and the level of consumption at any particular point of time. The height of the full employment growth path and consumption is affected by the savings ratio.

Simulation II, Solow Growth Model

Year	N	K	Q	I	Q/K	q	k	Q/L	Q
0	100	200	106	21	0.53			1.1	
1	102	221	110	22	0.50	4.0%	10.6%	1.1	981
5	110	315	128	26	0.41	3.5%	8.5%	1.2	858
10	122	451	150	30	0.33	3.2%	6.9%	1.2	773
25	164	1,004	227	45	0.23	2.6%	4.6%	1.4	700
50	269	2,545	410	82	0.16	2.3%	3.3%	1.5	806
75	442	5,245	704	141	0.13	2.1%	2.7%	1.6	1,085
100	724	9,811	1,179	236	0.12	2.1%	2.4%	1.6	1,569
150	1,950	29,888	3,195	639	0.11	2.0%	2.1%	1.6	3,642
200	5,248	83,526	8,470	1,694	0.10	2.0%	2.0%	1.6	9,012

Parameter values

λ	0.7
λ'	0.25
s	0.2
n	2.00%
ρ	0.05%
K(0)	200
N(0)	100
R	10
A	1.122018

Equilibrium Values

$q_e = k_e =$	1.93%
Q/Ke	0.097
K/Qe	10.3