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# CALCULUS REVIEW

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## 1. FUNCTIONS

**1.1. Definition:** A function *f* is a rule that associates each value of one variable with <u>*one and only one*</u> value of another variable.

For example, a function that associates with each value of *X*, one and only one value of *Y,* is written as

$$f: X \to Y$$

where *X* represents a set of values called the ***domain*** of the function *f*, and *Y* represents another set of values called the ***range*** of the function *f*. A standard way to represent this function is:

$$y = f(x)$$

where any one element *x* of the set *X* is associated to one and only one element *y* of the set *Y*. In this case ***y*** is called the ***dependent variable*** or the ***value of the function*** and ***x*** is called the ***independent variable*** or the ***argument of the function.*** The above function is also referred to as a ***univariate function***.

<u>***Example 1:***</u> Which of the following is a function?

(a) $y = 6x + 10$

(b) $x^2 + y^2 = 1$

### 1.2. The graph of a function

The ***coordinate system*** enables us to display two variables on a single graph. The graph of a function *f* consists of all points *(x,y)* where $y = f(x)$. To draw a graph, chose different values for *x* in the domain of the function and compute the corresponding $y = f(x)$. The value of *x* and the corresponding value of *y* are referred to as an ***ordered pair***. Construct a table of all ordered pairs.

We can graph these ordered pairs on a two-dimensional grid. The first number in each ordered pair, called the ***x-coordinate***, tells us the horizontal location of the point. The second number, called the ***y-coordinate***, tells us the vertical location of the point. The point with both an x-coordinate and a y-coordinate of zero is known as the ***origin***. The two coordinates in the ordered pair tell us where the point is located in relation to the origin: *x* units to the right of the origin and *y* units above it.

The points (if any) at which the graph crosses the $x$ and the $y$-axes are called the ***x and y intercepts***, respectively. To find the y-intercept of $y = f(x)$, set $x$ equal to 0 and compute $y$. Similarly, to find the $x$ intercept of $y = f(x)$, set $y$ equal to 0 and solve for $x$.

**1.3. Slope of a function:** shows how fast the dependent variable changes when the independent variable changes. The s*lope* of the function $y = f(x)$ is given by the following equation:

$$slope = \frac{\Delta y}{\Delta x} = \frac{f(x_B) - f(x_A)}{x_B - x_A}$$

where the ordered pairs ($x_A$, $f(x_A)$) and ($x_B$, $f(x_B)$) are two points on the function.

***Example 2:*** Find the slope of $y = 2 + 2x$.

***Result:*** If $y = a + bx$, where $a$ and $b$ are any numbers, the graph of the function is a straight line whose slope is $b$ and y-intercept is $a$.

A function of the type $y = a + bx$ is called ***linear*** because its graph is a straight line. More complicated functions, such as $y = x^2$, are ***nonlinear*** and their graphs are not straight lines. In such cases, it will matter which two points we pick to calculate the slope of the function. Rather than worry about which points we should pick, we usually draw a line which has the same slope as the function at the point we are interested in (which we call a *tangent line*) and calculate the slope of that line.

## 2. DERIVATIVES

The idea behind the derivative is to measure the slope of the function itself using two points which are very close together: as the points get arbitrarily close to one another, the slope of the line between those points will become the same as the slope of the tangent line, and will therefore become the slope of the function. As the distance between the points gets very close to 0, $\Delta x$ and $\Delta y$ will get close to zero, and we usually call them $dx$ and $dy$ instead (where the small $d$ indicates a very small difference.) We will therefore often refer to the derivative as $dy/dx$ instead of $\Delta y/\Delta x$ (although the idea is the same in both cases.)

### 2.1. Geometry of derivatives:

The slope of the secant line $AB$ represents the value of the difference quotient $\Delta y/\Delta x$ for an initial value of $x$ equal to $x_A$ and $\Delta x = x_B - x_A$. The slope of the secant line $AC$ represents the value of the difference quotient for the same initial value of $x$ and the smaller change in $x$, $\Delta x = x_C - x_A$. In the limit as $\Delta x$ approaches zero, the secant line becomes identical to the line that is tangent to the function at point $A$, which is depicted as the line $TAT$ in *Figure 1*. Thus the value of the derivative of the function at the point $x_A$, that is, $dy/dx$ evaluated at $x_A$, is the slope of the line tangent to the function at that point. Note that the value of the derivative varies with the point at which it is evaluated.

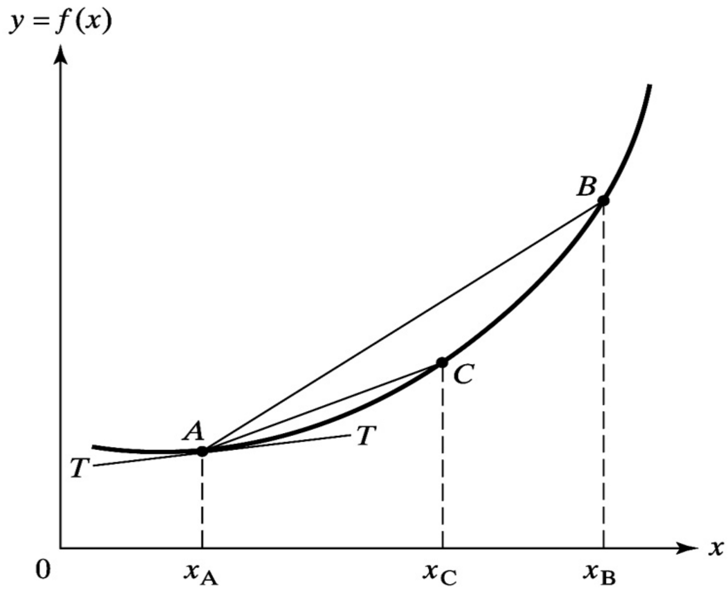


***Figure 1***

**2.2. Definition:** The derivative of the function $y = f(x)$, which is written as $dy/dx$, is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

provided that the limit exists.

Another way of denoting a derivative is by $f'(x)$.

***<u>Example 3:</u>*** Find the derivative of the quadratic function

$$y = a + bx_0 + cx_0^2$$

at $x_0$.

**2.3. Rules of Differentiation**

**Rule 1 (Scalar Rule):** If $f(x) = k$, where $k$ is a constant, then

$$f'(x) = 0.$$

***<u>Example 4:</u>*** If $y = 6$, what is the slope of the function?

**Rule 2:** If $f(x) = bx$ (a linear function), then

$$f'(x) = b.$$

**Rule 3 (Power Rule):** If $f(x) = k \cdot x^n$, then

$$f'(x) = n \cdot k \cdot x^{n-1}.$$

***Example 5:*** What is the derivative of

$$(a)\ y = x^2?$$

$$(b)\ y = x^{1/2}$$

**Rule 4 (Sum-Difference Rule):** For any two functions $f(x)$ and $g(x)$

$$\frac{d(f(x) \pm g(x))}{d(x)} = f'(x) \pm g'(x)$$

***Example 6:*** Find the derivative of $y = f(x) = x^3 - 5x + 6$

**Rule 5 (Product Rule):** For $f(x)=g(x)h(x)$,

$$f'(x)=g'(x)h(x) + h'(x)g(x)$$

***Example 7:*** Find the derivative of

$$y = (2x + 3)(3x^2)$$

**Rule 6 (Exponential Function Rule):** The derivative of the exponential function $f(x) = e^{kx}$ is

$$f'(x) = k \cdot e^{kx}$$

***Example 8:*** Find the derivative of

$$y = e^{x^2+3x+5}$$

**Rule 7 (Chain Rule):** The derivative of the composite function

$$y=f(x) = g(h(x))$$

where

$$u = h(x),$$

$$g(h(x)) = g(u)$$

And both $h(x)$ and $g(u)$ are differentiable functions, is

$$\frac{df(x)}{dx} = g'(h(x)) \cdot h'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$$

***Example 9:*** Find the derivative of

$$z = (x^2 + 3x - 2)^{17}$$

**Rule 8 (Quotient Rule):** The derivative of $f(x) = \frac{g(x)}{h(x)}$ is

$$f'(x) = \frac{g'(x) \cdot h(x) - h'(x) \cdot g(x)}{[h(x)]^2}$$

***Example 10:*** Find the derivative of

$$y = \frac{2x - 3}{x + 1}$$

**Rule 9 (Natural Logarithmic Function Rule):** The derivative of $f(x) = ln(x)$ is

$$f'(x) = \frac{d \ln x}{dx} = \frac{1}{x}$$

***Example 11:*** Find the derivative of

$$y = ln(x^3)$$

**2.4. Second Derivative**

The derivative of a function is itself a function. The derivative of the derivative of a function is that function's second derivative. The second derivative can be thought of as representing the rate of change of the rate of change of the original function.

***Definition:*** The second derivative of a function

$$y = f(x),$$

denoted as $f''(x)$ or $d^2y/dx^2$, is the derivative of its first derivative. That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The second derivative of a function is found by simply applying the rules of differentiation to the first derivative.

***Example 12:*** Find the second derivative of $f(x) = a\sqrt{x}$, where $a$ is a constant.

### 2.5. The Differential

The derivative of a function enables us to calculate the marginal rate of change of that function or the change in the dependent variable for an infinitesimal change in the independent variable. However, if we would like to estimate the overall change in the dependent variable that would arise in response to a given change in the independent variable, we have to use the differential.

***A. Definition:*** Define $dx$ as an arbitrary change in $x$ from its initial value $x_0$ and $dy$ as the resulting change in $y$ along the tangent line from the initial value of the function $y_0 = f(x_0)$. The differential of $y = f(x)$, $dy$, evaluated at $x_0$, is

$$dy = f'(x)\, dx$$

***B. Geometric Interpretation***

The differential, $dy$, shows the estimated change in $y$ due to a change in $x$ from $x_0$ to $x_0 + dx$, along a line tangent to the function at the point $x_0$. This estimated change in $y$ equals the product of $dx$ and the slope of the line tangent to the function at $x_0, f(x_0)$. In *Figure 2*, $dy = y_D - y_0$. The actual change in the value of the function, $\Delta y$, corresponding to this change in the argument of the function from $x_0$ to $x_0 + dx$ is

$$\Delta y = f(x_0 + dx) - f(x_0)$$

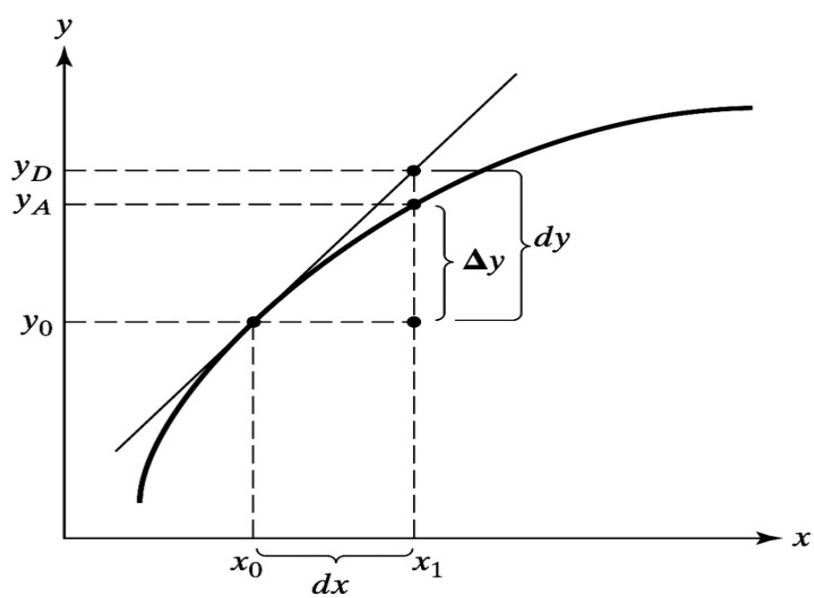


***Figure 2***

The actual change in the value of the function is depicted in the figure as the distance $y_A - y_0$. The differential $dy$ is not exactly equal to the actual change in the value of the function, $\Delta y$, though it does provide an approximation of the actual change.

***Example 13:*** Consider the relationship between import levels, *m*, and tariff rates, *t*. Suppose that these two variables are linked by the function $m = g(t)$ where

$$M = g(t) = 1000 - 200t + 250t^2$$

Suppose the initial tariff rate is 0.2 and it is scheduled to rise to 0.25. Find the estimated value of the change in imports using the differential.

## 3. SYSTEMS OF EQUATIONS

Economic models typically consist of a number of equations that represent identities, behavioral relationships, and conditions that constitute an equilibrium. These equations include both ***variables***, which are economic quantities that can assume different values, and ***parameters***, which are unvarying constants.

The variables of a model are classified as either ***exogenous*** if they are determined outside of the model or ***endogenous*** if they are determined by the model. A solution to the model is a representation of the endogenous variables as functions of only the parameters of the model and the exogenous variables.

***Example 14:***

$$\begin{aligned} Q^d &= \alpha - \beta P + \gamma G \\ Q^s &= \theta + \lambda P - \phi N \\ Q^d &= Q^S \end{aligned}$$

where $Q^d$ = *quantity of pizza demanded*
$Q^s$ = *quantity of pizza supplied*
$P$ = *price of pizza*
$G$ = *the price of a good which is a substitute of pizza*
$N$ = *the price of inputs used in producing pizza*

We can use the repeated substitution technique to solve this system of equations. The technique involves substituting out the endogenous variables such that you get an equation that contains only one endogenous variable.

## 4. MULTIVARIATE CALCULUS

4.1. A **Multivariate Function** has more than one variable as an argument. General form:

$$y = f(x_1, x_2, \ldots, x_n)$$

***Example 15:*** $C = 300 + 0.6I + 0.02W$
$C$ = *Consumption*
$I$ = *Income*
$W$ = *Wealth*

### 3.2. Graphs of bivariate functions (functions with two arguments)

It is not easy to graph a function of two variables. One way to visualize a function is to use level curves. To draw the level curves, fix some number, say *c* of the function *f* such that $c = f(x_1,x_2)$. The set of points $(x_1,x_2)$ in the $x_1$, $x_2$ plane that satisfy $c = f(x_1,x_2)$ is called the level curves of *f* at *c*.

Level curves appear in many economic models. For instance, you will see that indifference curves in consumer theory and isoquants in production theory are special level curves.

***<u>Example 16:</u>*** Draw the level curve of

$$z = x^{1/2} y^{1/2}$$

for $z = 5$ and *x* and *y* both positive.

### 3.3. Partial Derivatives

We are often interested how the value of the multivariate function changes in response to a very small change in one of its arguments.

**Definition:** The partial derivative of the function $y = f(x_1, x_2, \ldots, x_n)$ with respect to its argument $x_i$, written as $\partial y / \partial x_i$, is

$$\frac{\partial y}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x_1,\ldots,x_i + \Delta x_i,\ldots,x_n) - f(x_1,\ldots,x_i,\ldots,x_n)}{\Delta x_i}$$

provided that the limit exists.

The partial derivative of a multivariate function with respect to one of its arguments is found by applying the rules for univariate differentiation and <u>*treating all the other arguments of the function as constant*</u>.

***Example 17:*** The Cobb-Douglas production function is widely used in economics. It is given by

$$Q = AK^{1-\alpha} L^{\alpha}$$

where *Q = quantity of output*
*K = amount of capital*
*L = amount of labor*
*A = measure of productivity*
$0 < \alpha < 1$

Find the marginal product of labor, which is found by taking the partial derivative of output with respect to labor.

### 3.4. Total differential

Similar to the total differential of a univariate function, the total differential of a multivariate function shows how small changes in all the arguments of a function affect the value of the function.

**Definition:** The total differential of the multivariate function

$$y = f(x_1, x_2, \ldots, x_n)$$

evaluated at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ is

$$dy = f_1(x_1^0, x_2^0, \ldots, x_n^0)dx_1 + f_2(x_1^0, x_2^0, \ldots, x_n^0)dx_2 + \ldots + f_n(x_1^0, x_2^0, \ldots, x_n^0)dx_n$$

where $f_i(x_1^0, x_2^0, \ldots, x_n^0)dx_i$ represents the partial derivative of the function $f(x_1, x_2, \ldots, x_n)$ with respect to its *i-th* argument, evaluated at the point $(x_1^0, x_2^0, \ldots, x_n^0)$.

***<u>Example 18:</u>*** Consider the function

$$w = 2x^2 + \frac{1}{2}xy - 3y^3$$

What is the total differential of the function?

## 5. OPTIMIZATION

A solution to many types of economic models requires the identification of an optimal outcome. The basic model of the firm, for example, is based on the assumption that the goal of the firm is to achieve the highest level of profits. The solution to the problem facing the firm is to determine the point where the profit function achieves a maximum.

The largest value of a function over its entire range is called its ***global (or absolute) maximum***, and the smallest value of a function over its entire range is called its ***global (or absolute) minimum***. The largest value within a small interval is called ***a local (or relative) maximum.*** The smallest value within a small interval is called ***a local (or relative) minimum***.

The derivative $f'(x_0)$ shows the change in the variable $y$ relative to some small change in the variable $x$ at point $x_0$. If the derivative is positive, the function is increasing at that point. If it is negative, the function is decreasing. Therefore, the slope of the function must equal zero at a point at which the function reaches a (local) maximum. This observation is the basis for our first optimization rule:

**First-order condition for an optimum:** The function $y = f(x)$ is at a local maximum or minimum at points at which $dy/dx = f'(x) = 0$.

Notice that this rule does not provide a criterion to distinguish between a maximum or a minimum. To do this, we need an additional definition.

**Second-order condition for an optimum**: If $f'(x) = 0$ at some point, then this point is a local maximum if $f'(x) < 0$. It is a local minimum if $f'(x) > 0$.

If the second derivative is positive, the derivative is increasing; if it is negative, the derivative is decreasing. The slope of a function must be decreasing around a point at which the function has zero slope if the function is to be at a maximum. Similarly, the slope of a function must be increasing around a point at which the function has zero slope if the function is to be at a minimum.

***<u>Example 19:</u>*** Suppose you are given the following function:

$$f(x) = 100x - x^2$$

At what value of $x$ is the function maximized?

***<u>Solution:</u>***

$$f'(x) = 100 - 2x = 0$$

Solving this equation for $x$ tells us that the function is maximized at $x^* = 50$.

To check whether this is a maximum or a minimum, take the second derivative of the function:

$$f'(x) = -2 < 0$$

Therefore, $x^* = 50$ is a maximum.

Let’s plot the graph of *f*.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ***x*** | -20 | -10 | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
| ***f(x)*** | -2400 | -1100 | 0 | 900 | 1600 | 2100 | 2400 | 2500 | 2400 |

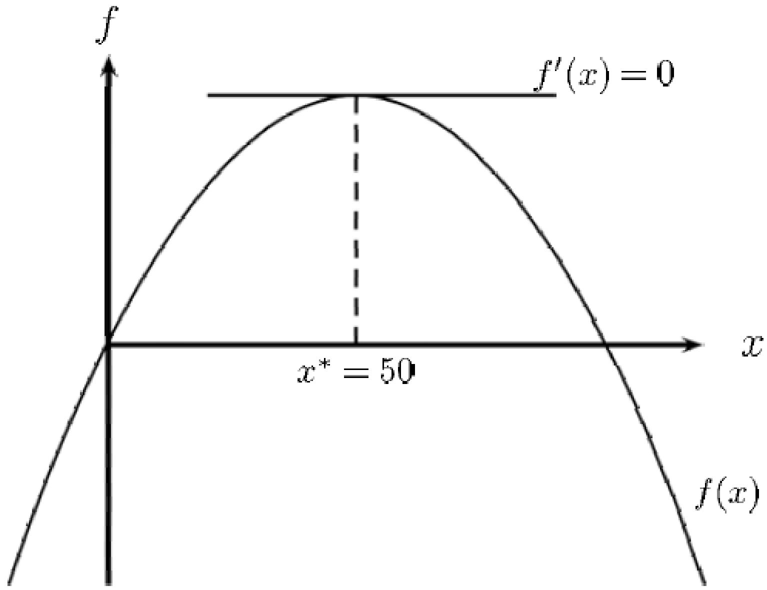


***Figure 3***

The graph of the function has a maximum where the slope of the tangent line is zero. In other words, *f(x)* is maximized at the value that satisfies *f'(x)* = 0.

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