

# **Introduction to Calculus for Business and Economics**

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# Introduction to Calculus for Business and Economics

## I. Functions

$y = f(x)$  is a function of  $x$  if and only if, for each  $x$  in the domain of  $f(x)$ , that is the values of  $x$  for which  $f(x)$  is defined, there is exactly one value of  $f(x)$ .

Examples:

1.  $y = 2 - 3x$  is a function
2. The real values of  $y$  satisfying the equation  $x^2 + y^2 = 1$ ,  $x \in [-1, 1]$ , are not a function of  $x$ . (This is the unit circle at  $(0,0)$ )

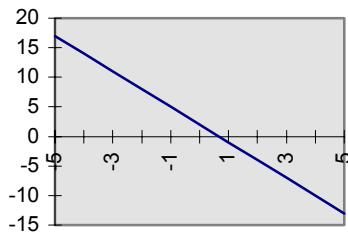


Figure 1.  $y = 2 - 3x$

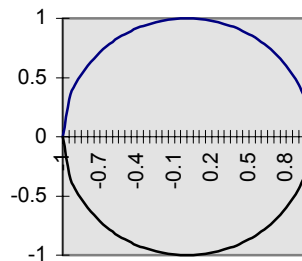


Figure 2. The unit circle

## II. Limits and rules

Look at  $y = f(x)$  in Figure 3 below.

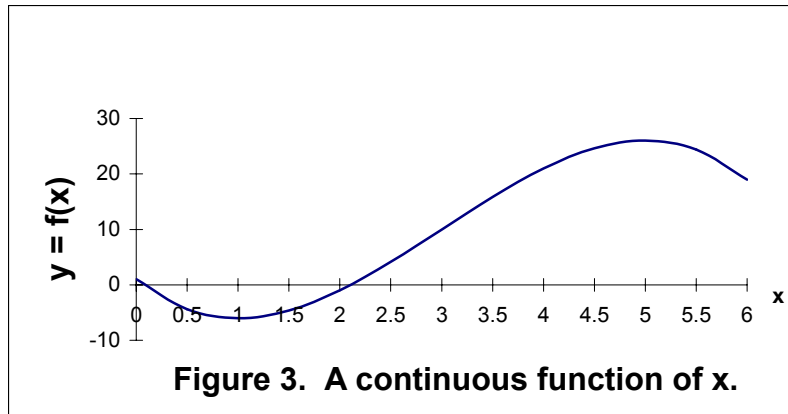


Figure 3. A continuous function of  $x$ .

We see  $f(4) = 21$ . Thus, we see that as  $x$  approaches 4,  $f(x)$  approaches 21. In general, if  $f(x)$  has a limit at  $x = a$ , then the limit of  $f(x)$  as  $x$  approaches the value  $a$  equals the value of  $f(x)$  at  $x = a$ . We write this as  $\lim_{x \rightarrow a} f(x) \equiv f(a)$ .

If this is true for all  $x$  in the domain of  $f(x)$ , we say that  $f(x)$  is continuous. Sometimes this does not occur. For example the relationship between the weight of a letter and the postage required is discontinuous at each ounce. Up to and including one ounce the postage is 32 cents. Greater than one ounce up to and including two ounces the postage is 50 cents (sic), etc. In this case the limits differ depending upon the direction you come from;  $f(1) =$  limit from below (32 cents), but not the limit from above (50 cents).

If  $f(x)$  and  $g(x)$  are functions of  $x$ , then  $f(x) \pm g(x)$  are also functions of  $x$ , as is  $f(x) \cdot g(x)$ . Also,  $f(x)/g(x)$  is a function of  $x$  for all  $g(x) \neq 0$ . In addition, if

$\lim_{g(x) \rightarrow 0} f(x)/g(x)$  exists, then  $f(x)/g(x)$  is also a function of  $x$ .

Furthermore, the fact that  $\lim_{x \rightarrow a} f(x) = f(a)$  also means that the following hold:

$$\lim_{x \rightarrow a} f(x) \pm g(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = f(a) \pm g(a),$$

$$\lim_{x \rightarrow a} f(x) \cdot g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = f(a) \cdot g(a), \text{ and}$$

$$\lim_{x \rightarrow a} f(x) / g(x) = \lim_{x \rightarrow a} f(x) / \lim_{x \rightarrow a} g(x) = f(a)/g(a), \text{ where } g(a) \neq 0.$$

### **III. The derivative; maxima, minima, and points of inflection**

One very important application of the quotient property above is the special limit known as the derivative function. In Figure 3 above, we saw that  $f(4) = 21$ . (The actual equation used was  $y = -x^3 + 9x^2 - 15x + 1$ ) We also see that  $f(5) = 26$ . The slope of the line joining the two points (4,21) and (5,26) is found by taking  $\Delta y / \Delta x = (26-21)/(5-4) = 5$ . Now what happens as we calculate the following limit?

$$\Delta y / \Delta x = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ as } \Delta x \rightarrow 0.$$

The slope of the line joining the two points becomes the slope of the curve at  $(x, f(x))$ , that is, the slope of the line tangent to  $f(x)$  at  $x$ . This limit is called the derivative of  $f(x)$  at  $x$ , and is written as either  $f'(x)$  or  $dy/dx$ . We can also define this limit as follows:

Let  $y = f(x)$  be continuous on the interval  $[a,b]$ ,  $x_0, x_0 + \delta$  both  $\in [a,b]$ .

$$\text{Then } dy/dx = \lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta}.$$

Now we notice that the denominator of this function equals 0 at the limit. This would tend to indicate that this function is undefined (ie.  $= \infty$ ) at this point. But the numerator

also is identically equal to zero at  $\delta = 0$ , so the function is indeterminate rather than undefined. To evaluate the function at the limit point, we need to calculate the limit itself. Let us look at some examples.

Example 1: Let  $y = f(x) = a + bx$ , which we learned is a straight line with intercept  $a$  and constant slope  $b$ .

$$\text{Then } dy/dx = \lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta} = \lim_{\delta \rightarrow 0} \frac{(a + b(x + \delta)) - (a + bx)}{\delta}$$

$$= \lim_{\delta \rightarrow 0} \frac{b\delta}{\delta} = b.$$

Example 2: Let  $y = f(x) = ax^2$ .

$$\lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta} = \lim_{\delta \rightarrow 0} \frac{a(x + \delta)^2 - ax^2}{\delta} = \lim_{\delta \rightarrow 0} \frac{ax^2 + 2ax\delta + \delta^2 - ax^2}{\delta}$$

$$\lim_{\delta \rightarrow 0} \frac{2ax\delta + \delta^2}{\delta} = 2ax.$$

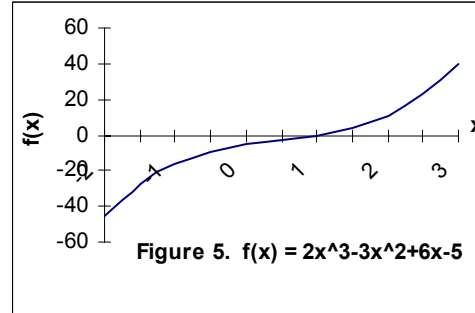
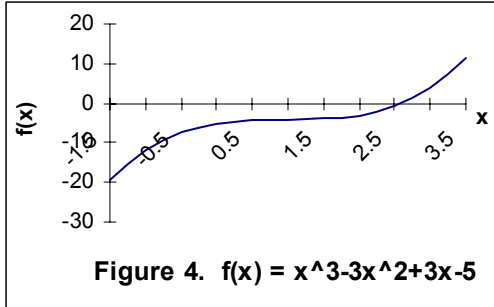
Example 3: Let  $y = f(x) = ax^n$ , where  $n$  is a positive integer. Now  $f(x + \delta) = a(x + \delta)^n = a(x^n + nx^{n-1}\delta + n(n-1)/2 \cdot x^{n-2}\delta^2 + \dots + n!/(k!(n-k)!)x^{n-k}\delta^k + \dots + \delta^n)$ . Thus,  $(f(x + \delta) - f(x))/\delta = anx^{n-1} + \Sigma$  other terms with multiples of  $\delta$ . These later terms all equal 0 at  $\delta = 0$ . Thus,  $dy/dx = anx^{n-1}$ . It is also the case for all  $n$ , whether a positive integer or not.

Example 4. Now, from the additive property of limits, it follows that if  $y = f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_kx^k + \dots + a_0$ , then

$dy/dx = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + ka_kx^{k-1} + \dots + a_1$ . Thus, the derivative of a polynomial of degree  $n$  is a polynomial of degree  $n-1$ .

Let us return to the function in Figure 3,  $f(x) = -x^3 + 9x^2 - 15x + 1$ . The derivative is  $dy/dx = -3x^2 + 18x - 15$ . This equation provides us with the slope of  $f(x)$  at each  $x$ . Now, the slope at any maximum point or minimum point obviously is zero. So we can set  $dy/dx = 0$ , solve the resulting polynomial for  $x$  and find these points. In this case we get

$-3x^2 + 18x - 15 = 0 \Rightarrow x^2 - 6x - 5 = 0$ ;  $\therefore x = 5, 1$  are the solutions. Indeed, we see that  $f(x)$  reaches a maximum at  $x = 5$  and is minimized at  $x = 1$ . Since a quadratic function has at most two real solutions, there are no other turning points. In fact, a quadratic equation may have no, one (double), or two solutions. The one we just solved had two, and therefore a maximum and a minimum. Figures 4 and 5 below show the cases of a double solution and no solution, respectively. Solving the first,  $f(x) = x^3 - 3x^2 + 3x - 5$ , gives



$3x^2 - 6x + 3 = 0$ ; or  $x = 1, 1$ . This is a point of inflection with slope = 0 at  $x = 1$ . An inflection point is one at which the slope stops increasing and begins to decrease, or vice versa. The second function,  $f(x) = 2x^3 - 3x^2 + 6x - 5$ , yields  $dy/dx = 6x^2 - 6x + 6 = 0$ , with no real roots; we note, however, that there is a point of inflection at  $x = 1$ .

Points of inflection are points at which the *slope* achieves a maximum or a minimum; thus, *the change in the slope is zero*. This means the second derivative of the original function is zero.

The second derivative is useful in another capacity. Suppose we find a turning point. We cannot be sure whether this point is a maximum or a minimum. But the value of the second derivative at the  $x$ -value will tell us which it is. If the value of the second derivative is negative, this implies that the slope is increasing; this can only be the case if the slope is changing from negative to positive, which occurs at a minimum point. Similarly, if the second derivative is positive, the function must be at a maximum. Finally, if the second derivative is zero, again we are at an inflection point with zero slope.

For the function in Figure 4 above, the value of the second derivative at  $x = 1$  is zero, so the point is an inflection. The function in Figure 3 was  $f(x) = -x^3 + 9x^2 - 15x + 1$ . The first derivative is  $dy/dx = -3x^2 + 18x - 15$ . Thus, we solve  $x^2 - 6x + 5 = 0$ , for roots, to get  $x = 1$  and  $5$ . The second derivative is  $d^2y/dx^2 = -6x + 18$ , which is negative at  $x = 5$  and positive at  $x = 1$ . Thus, we have a minimum at  $x = 1$  and a maximum at  $x = 5$ .

#### **IV. Functions of more than one variable, partial derivatives, and the Lagrangean**

In the real world we often find that the value of a variable depends on more than one other variable. For example, consumers derive satisfaction, or utility, from the consumption of many different goods. Level of production depends upon the levels of inputs of many different factors: labor, capital, materials, etc. To take into account that the level of output  $Y$  depends upon the levels of inputs of several factors,  $X_1, X_2, \dots, X_n$ , we will write this as  $Y = f(X_1, X_2, \dots, X_n)$ .

We are often interested in the effect of a one unit change in one of the inputs, say input  $i$ . Will write this partial derivative as  $\partial Y / \partial X_i = f_i(X_1, X_2, \dots, X_n)$ . Let us show how this is done for a few cases.

Example 1. Let  $f(X_1, X_2) = 2X_1 - 3X_2$ . The  $f_1 = 2$  and  $f_2 = -3$ , since we assume in each case that the level of the other good is unchanged, that is, it is constant.

Example 2. Let  $f(X_1, X_2) = 2X_1X_2$ . Then  $f_1 = 2X_2$  and  $f_2 = 2X_1$ .

Example 3. Let the output of widgets depend upon the inputs of labor  $L$  and capital  $K$  as follows:  $Q = 2L^4K^6$ .

Then  $Q_L = \partial Q / \partial L = 2(4)K^6L^{4-1} = .8K^6L^{-6} = .8(K/L)^6$ . This implies that the more labor currently in use the *less* productive each unit of labor and the *more* productive each unit of capital. Thus, if you give a worker more capital to use, the more productive that worker is.  $\partial Q / \partial L$  is called the marginal product of labor. Similarly, the marginal product of capital is found to be  $Q_K = 2(6)L^4K^{6-1} = 1.2(L/K)^4$ ; that is, the marginal product of capital is greater the more labor used and smaller the more capital used.

Example 4. Let  $l$  = length of a rectangle,  $w$  = the width. How much does the area of the rectangle change if we increase the width one unit? The length one unit? For any given perimeter, what is the maximum area achievable?

Area  $A = LW$ , so  $\partial A / \partial W = L$ ,  $\partial A / \partial L = W$ . Thus if  $W = 10''$ ,  $L = 20''$ ,  $\partial A / \partial W = 20$  sq." and  $\partial A / \partial L = 10$  sq.".

The perimeter of a rectangle  $P = 2L + 2W$ , so  $W = P/2 - L$ . Thus,  $A = LW = L(P/2 - L) = PL/2 - L^2$ . To maximize  $A$  we set  $dA/dL = 0$ .  $dA/dL = P/2 - 2L = 0$ , so  $L = P/4$ . That is  $L = W$  and we have a square. Note, in this case we did not need to use the partial derivative as we substituted the value of  $W$  in terms of  $P$  and  $L$ .

We could not have solved the problem differently by solving  $\partial A / \partial L = 0$ , and  $\partial A / \partial W = 0$  simultaneously and then used the relationship between  $P$ ,  $L$  and  $W$ , since we would have gotten the optimal solution by setting both  $L$  and  $W$  equal to zero. We did not impose on the solution that the perimeter not equal zero.

There is a method that allows us to impose restrictions on variables while optimizing the values of a function of these variables. This method is known as the Lagrangean method and requires the use of a dummy function called the lagrangean. In the simple example above, we are trying to maximize the objective function  $A = LW$ , subject to the condition that  $L + W = P/2$ , where  $P$  is some number  $> 0$ . This constraint can be written as  $P/2 - L - W = 0$ .

We let the lagrangean  $\mathbf{L} = LW - \lambda(P/2 - L - W)$ .  $\lambda$  is known as the lagrangean multiplier and becomes another variable in the equation. We now maximize  $\mathbf{L}$  with respect to the three variables of the equation,  $L$ ,  $W$ , and  $\lambda$ .

$$(1) \partial \mathbf{L} / \partial L = W - \lambda(-1) = W + \lambda = 0$$

$$(2) \partial \mathbf{L} / \partial W = L - \lambda(-1) = L + \lambda = 0$$

$$(3) \partial L / \partial \lambda = -(P/2 - L - W) = 0$$

Solving (1) and (2) we get that  $W = L = -\lambda$ . (3) tells us that  $L + W = P/2$ ;  $\therefore 2L = P/2$ . So  $L = W = P/4$ .

### V. The total differential

The last topic of differential calculus is the so-called total differential function. Let us once again look at the Area function,  $A = LW$  and its partials  $\partial A / \partial W = L$ ,  $\partial A / \partial L = W$ .

$\partial A / \partial W = L$  means that  $dA = L dW$  if  $L$  is held constant and  $\partial A / \partial L = W$  tells us that  $dA = W dL$  when  $W$  is fixed. If both  $L$  and  $W$  vary at the same time the new area is given by  $(L + dL)(W + dW) = LW + L dW + W dL + dL dW$ . The total effect on  $A$  can be approximated, for only small changes in  $L$  and  $W$ , by  $dA = (\partial A / \partial W) dL + (\partial A / \partial L) dW = L dW + W dL$ .

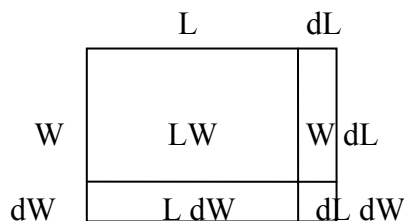


Figure 6. The area of a rectangle

If  $L = 20''$ ,  $W = 10''$  and  $dL = dW = .1''$ , then  $dA \approx 20''(.1'') + 10''(.1'') = (2 + 1) \text{ sq.}'' = 3 \text{ sq.}''$ . To check, we calculate the actual area before and after the changes.  $LW = 20(10) = 300 \text{ sq.}''$  before, and  $(20.1)(10.1) = 203.01 \text{ sq.}''$  after the changes. Thus, our estimated change of 3 was approximately correct. The reason we were off is that we omitted the  $dL dW = (.1)(.1) = .01$ . If the changes are small relative to the original size of the variables, the error will be relatively small. These additional effects are often referred to as second order effects and, if these are small, may be ignored.

Another useful application of the total differential approach is to estimate the effect on total product of an increase in the amount of labor and capital in the production function we studied earlier,  $Q = 2L^4K^6$ , where  $L$  is in hours of labor and  $K$  is the number of machines.

$$\text{Then } dQ \text{ is given by } Q_L dL + Q_K dK = .8(K/L)^6 dL + 1.2(L/K)^4 dK.$$

Suppose initially  $L = 100,000 \text{ hrs.}$ ,  $K = 32 \text{ machines}$ . Then  $Q = 2(100,000)^4(32)^6 = 2(100)(8) = 1600 \text{ widgets}$ . Now let  $dL = 100 \text{ hrs.}$ ,  $dK = .01 \text{ machines}$ .

$$dQ = .8(32/100,000)^6(100) + 1.2(100,000/32)^4(.01) = .8(8/1000)(100) + 1.2(100/4)(.01) = 6.4 + 0.3 = 6.7 \text{ widgets. To check:}$$

$$Q_2 = 2(100,100)^4(32.01)^6 = 1606.682 \approx \text{increase of 6.7 widgets from 1600.}$$

## VI. Integral calculus

Elementary calculus consists of two parts: differential calculus, covered in sections I. Through V., and integral calculus. Integral calculus is closely related to differential calculus, as we shall see shortly, but deals with a different issue, namely, the determination of the area under a curve.

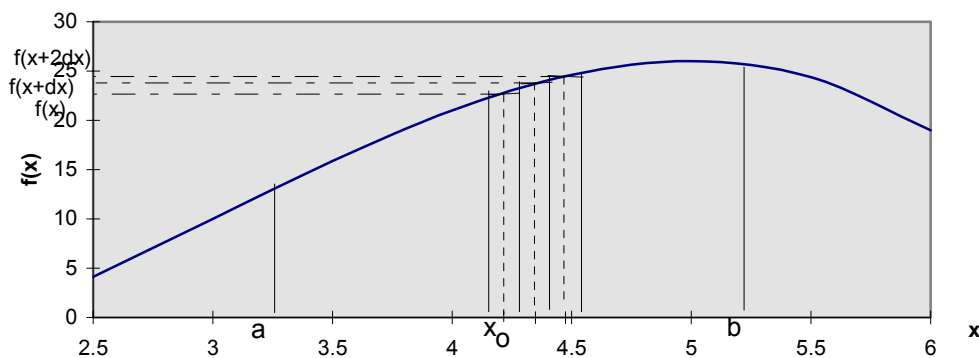


Figure 7.  $y = f(x)$

Suppose we want to find the area under the curve  $y = f(x)$  between  $x = a$  and  $x = b$  in Figure 6 above. We could approximate that area by finding the areas of a series of very narrow rectangles, say of width  $\Delta x$ , from  $a$  to  $b$ . We see three such rectangles starting with the one centered at point  $x = x_0$  on the figure. The area of this rectangle is approximately  $f(x_0) \cdot \Delta x$ . The next rectangle has an area of approximately  $f(x_0 + \Delta x) \cdot \Delta x$  and the third rectangle's area is about  $f(x_0 + 2\Delta x) \cdot \Delta x$ . We could thus calculate the area from  $a$  to  $b$  by taking the sum

$$\sum_{x=a}^{x=b} f(x + \Delta x) \cdot \Delta x.$$

In order to improve on the accuracy of the estimate, all we need to do is let the rectangles be narrower and narrower, that is, have  $\Delta x \rightarrow 0$ . Thus, the exact area is given as

$$\lim_{\Delta x \rightarrow 0} \sum_{x=a}^{x=b} f(x + \Delta x) \cdot \Delta x.$$



This limit is written as  $\int_a^b f(x) dx$ , called the integral of  $f(x)$  from  $x = a$  to  $x = b$ . Now, let  $F(x)$  be the general integral of  $f(x)$ . What is the functional form of  $dF(x)/dx$ ? This is the change in the area function  $F(x)$ . But the change of the area is the height of the function, which is just  $f(x)$  itself. This means that the integral of  $f(x)$  is merely the anti-derivative of  $f(x)$ ; that is, the function whose derivative is  $f(x)$ . It should be noted that the exact function is unknown to within a constant of integration. That is because the derivative of a constant is zero.

Example. To find the area under the function  $f(x) = 2x - 3$  between  $x = 2$  and  $x = 5$ , we first find the function  $F(x)$  that has the derivative  $2x - 3$ . We know it is a polynomial of degree 2; in fact,  $F(x) = x^2 - 3x + c$ , where  $c$  is any constant, has the desired derivative. Now since the lower limit of possible values of  $x$  is 2,  $F(2)$  must equal 0.  $F(2) = 2^2 - 3(2) + c = 0 \Rightarrow c = 6 - 4 = 2$ . Thus,  $F(x) = x^2 - 3x + 2$  and  $F(5) = 25 - 3(5) + 2 = 12$ . To check, we calculate the area of the triangle and the rectangle that lie under the curve between 2 and 5. Since  $f(2) = 1$  and  $f(5) = 7$ , the area of the triangle is  $\frac{1}{2}bh = \frac{1}{2}(5-2)(7-1) = 9$ ; the area of the rectangle is  $(5-2)(1-0) = 3$ ; and  $9 + 3 = 12$ .

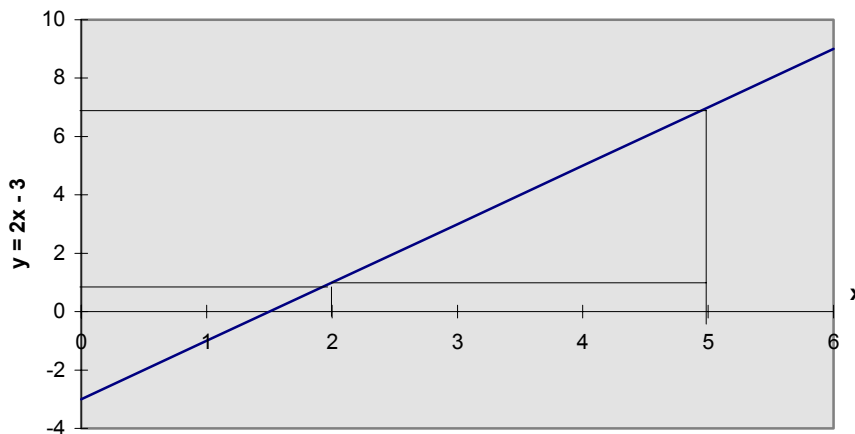


Figure 8.  $f(x) = 2x - 3$

Appendix: The derivatives and integrals of common functions.

Function	Derivative	Integral
$e^x$	$e^x$	$e^x$
$e^{rx}$	$re^{rx}$	$(1/r)e^{rx}$
$a^x$	$\ln(a)a^x$	$a^x/\ln(a)$
$\ln(x)$	$1/x$	$x(\ln(x)-1)$
$\sqrt{x}$	$1/2 \sqrt{x}$	$(2/3)x^{3/2}$
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$