# A Simple Proof of the FWL (Frisch-Waugh-Lovell) Theorem*

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Ragnar Frisch and F. V. Waugh (1933) demonstrated a remarkable property of the method of least squares in a paper published in the very first volume of *Econometrica*. Suppose one is fitting by least squares the variable $Y_t$ on a set of $k'$ explanatory variables plus a linear time trend, t = 1, 2, …

$$Y_t = b_0 + b_1 X_{1t} + b_2 X_{2t} + \ldots + b_{k'} X_{k't} + dt + e_t \tag{1}$$

As an alternative to the direct application of least squares, they considered the following two-step trend removal procedure:

*Step 1:* Detrend all the $X_{it}$ and $Y_t$ by first regressing each on the time variable,

$$X_{it} = c_{i0} + c_{i1}t + e^x_{it} \text{ ,and} \tag{2}$$

$$Y_t = c_0 + c_1 t + e^y_t \,, \tag{3}$$

and using the residuals from these least-squares regressions to calculate the detrended variables,

$$X^*_{it} = \overline{X}_i + e^x_{it} \,,\ i = 1, \ldots,\ k' \,,\ \text{and} \tag{4}$$

$$Y^*_t = \overline{Y} + e^y_t \,. \tag{5}$$

*Step 2:* Run the detrended regression:

$$Y^*_t = b^*_0 + b^*_1 X^*_{1t} + b^*_2 X^*_{2t} + \ldots + b^*_{k'} X^*_{k't} + e^*_t \,. \tag{6}$$

Frisch and Waugh proved a surprising proposition:

> Exactly the same coefficients are obtained with regression (6), based on detrended variables, as with regression (1), which includes trend as an explanatory variable; i.e., $b^*_i = b_i$, for i = 0,…, $k'$.

It is important to note that the fact that the least squares regression coefficients $b_i$ and $b^*_i$ are identical means that neither is superior to the other as an estimator of the unknown parameters $\beta_i$ of the underlying stochastic process that may be generating the data. It is also true that the $e_t = e^*_t$, which obviously means that examining either set of residuals will convey precisely the same information about the properties of the unobservable stochastic disturbances $\varepsilon_t$.

Lovell (1963) generalized their result by showing that the same regression coefficients will be obtained not just with a trend variable but with seasonal variables or indeed *any* non-empty subset of the explanatory variables in a regression. This result is variously known as the "FWL," the "Frisch-Waugh-Lovell," the "Frisch-Waugh" or the "decomposition" theorem.

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# The FWL Theorem

Suppose we partition the explanatory variables of a k variable multiple regression into any two non-empty sets, one consisting of $k'$ variables $X_{it}$ on which our attention is primarily focused and the other a set of $k'' = k - k'$ auxiliary variables $D_{it}$ :[1]

$$Y_t = b_1 X_{1t} + b_2 X_{2t} + \ldots + b_{k'} X_{k't} + d_1 D_{1t} + d_2 D_{2t} + \ldots + d_{k''} D_{k''t} + e_t . \tag{7}$$

Now consider the alternative least-squares regression equation:

$$Y_t^* = b_1^* X_{1t}^* + b_2^* X_{2t}^* + \ldots + b_{k'}^* X_{k't}^* + e_t^* \tag{8}$$

Here the $Y_t^*$ and $X_{it}^*$ are "cleansed" values of the dependent variable and the focus subset of the explanatory variables:

$$Y_t^* = \overline{Y} + e_t^y \text{ and} \tag{9}$$

$$X_{it}^* = \overline{X}_i + e_{it}^x ,\ i = 1 \,..\, k' ,$$

where $e_t^y$ and the $e_{it}^x$ are the least squares residuals obtained from the auxiliary regressions

$$Y_t = a_{y1} D_{1t} + \ldots + a_{yk''} D_{k''t} + e_t^y \tag{10}$$

$$X_{it} = a_{i1} D_{1t} + \ldots + a_{ik''} D_{k''t} + e_{it}^x ,\ i = 1 \,..\, k' . \tag{11}$$

Then:

$$b_i^* = b_i \quad \text{for i = 1,.., } k' \text{ and} \tag{12}$$

$$e_t^* = e_t . \tag{13}$$

Frisch and Waugh had employed Cramer's Rule in proving their trend theorem whereas Lovell (1963, 1007-8) used matrix algebra in establishing the more general FWL Theorem. Davidson and MacKinnon (1999, 62-9) presented both a geometric demonstration and a matrix proof of the result in their econometrics textbook; Green (2003, 26-7) and Johnston and Dinardo (1997, 101-3) employed matrix algebra in their texts.

In this note I will use simple algebra in showing how the FWL theorem can be easily derived from two well-known numerical properties of the method of least squares:

Property 1. The residuals from a least squares regression are uncorrelated with the explanatory variables.

Property 2. The coefficients of a subset of the explanatory variables in a regression equation will be zero if those variables are uncorrelated with both the dependent variable and the other explanatory variables.[2]

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[1] To ease notation I adopt the standard convention of subsuming the intercept in with the other explanatory variables by setting all values of an additional explanatory variable identically equal to one.

[2] This is easily seen in the simplest case of only two explanatory variables: the first multiple regression coefficient, given the presence of $x_2$, is $b_{1.2} = (\sum yx_1 \cdot \sum x_2^2 - \sum yx_2 \cdot \sum x_1 x_2) / \left[ \sum x_1^2 \cdot \sum x_2^2 - (\sum x_1 x_2)^2 \right]$ ,which reduces to $b_1 = \sum yx_1 / \sum x_1^2$ if $\sum x_1 x_2 = n s_{x_1} s_{x_2} r_{12} = 0$ ; if in addition $r_{x_1 y} = 0$, then $b_1 = 0$.

## Proof:

Substituting (10) into (7) yields

$$\begin{aligned} e_t^y = b_1 e_{1t}^x + \ldots + b_{k'} e_{k't}^x + (b_1 a_{11} + \ldots + b_{k'} a_{k'1} + d_1 - a_{y1}) D_{1t} + \ldots \\ + (b_1 a_{1k''} + \ldots + b_{k'} a_{k'k''} + d_{k''} - a_{yk''}) D_{k''t} + e_t \,. \end{aligned} \tag{14}$$

Because auxiliary equations (10) are fitted by the method of least squares, Property 1 implies that the residuals $e_{it}^x$ and $e_t^y$ from those regressions are uncorrelated with the $D_{it}$ explanatory variables. Therefore, all the regression coefficients of the $D_{it}$ in (14) are zero, thanks to Property 2, which means that precisely the same $b_i$ are obtained when the $D_{it}$ are dropped from the regression; that is,

$$e_t^y = b_1 e_{1t}^x + b_2 e_{2t}^x + \ldots + b_{k'} e_{k't}^x + e_t \,. \tag{15}$$

Adding the identity $\overline{Y} = b_1 \overline{X}_1 + b_2 \overline{X}_2 + \ldots b_{k'} \overline{X}_{k'}$ to (15) yields

$$\overline{Y} + e_t^y = b(\overline{X}_1 + e_{1t}^x) + b_2 (\overline{X}_2 + e_{2t}^x) + \ldots + b_{k'} (\overline{X}_{k'} + e_{k't}^x) + e_t \,, \tag{16}$$

which by (9) is equation (8), thus establishing that the least square coefficients $b_i^*$ of equation (8) are identical to the $b_i$ of equation (7) and that $e_t^* = e_t$.

# COMMENTS

1. There are $n - k < n - k'$ degrees of freedom in regressions (8) and (15) as well as (7). Therefore, execution of either regression (8) or (15) with a standard least-squares regression computer program neglecting this complication will yield too small a value for the standard error of the estimate, $\overline{S}_e$, and exaggerated t and p-values for the regression coefficients (Lovell 1963, 1002-3).
2. Because the least squares residuals calculated with regressions (7) and (8) are identical, precisely the same Durbin-Watson statistics will be generated.
3. The application of Aitkens Generalized Least Squares to regression equation (8) or (15) will result in less efficient estimates than its direct application to regression (7) (Lovell 1963, 1004).
4. Precisely the same regression coefficients but different residuals are generated when $Y_t$ instead of $Y_t^*$ is used as the dependent variable in (8) (Lovell 1963, 1001).

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