Slutzky Equation Example

Suppose that utility is

$$U = \sqrt{XY} \tag{1}$$

Then maximizing utility subject to the income constraint $M = p_x X + p_y Y$ yields the demand equation

$$X(M,p_x,p_y) = \frac{M}{2p_x} \tag{2}$$

Similarly, $Y(M,p_x,p_y) = M/2p_y$. Substituting these two demand functions back into the utility function yields the *indirect utility function* showing utility as a function of income and prices:

$$U^I(M,p_x,p_y) = \frac{M}{2\sqrt{p_x p_y}} \tag{3}.$$

$\partial U^I(M,p_x,p_y)/\partial M$ is called the *marginal utility of money*. It turns out that for any utility function the Lagrangian multiplier $\lambda = \partial U^I(M,p_x,p_y)/\partial M$.[1] Thus we have an economic interpretation of the Lagrangian multiplier.

Simple algebra now yields the *expenditure function* showing the income required to obtain a specified level of utility, given prices:

$$M(U,p_x,p_y) = 2U\sqrt{p_x p_y} \tag{4}$$

Substituting (4) into (2) yields the *compensated demand function* showing the demand for X given the specified level of utility and prices, which is relevant for a worker whose wages are accurately indexed (adjusted for inflation):

$$X^c(U, p_x,p_y) = 2U\sqrt{p_x p_y}\,/2p_x = U\sqrt{\frac{p_y}{p_x}}\,. \tag{5}$$

To check out the Slutzky equation, we compute its three key components:

The *total effect* of the price change: $\partial X(M,p_x,p_y)/\partial p_x = -\frac{M}{2p_x^2}$ (6)

The *income effect* of the price change: $-X\,\partial X(M,p_x,p_y)/\partial M = -\frac{X}{2p_x} = -\frac{M}{4p_x^2}$ (7)

The *substitution effect* $\partial X^c(U, p_x,p_y)/\partial p_x = -\frac{U\sqrt{p_y}}{2p_x^{3/2}} = -\frac{U\sqrt{p_x p_y}}{2p_x^2} = -\frac{M}{4p_x^2}$ (8)

As always, for *any* utility function, the total effect is the sum of the income effect plus the substitution effect. However, that a slightly different notation is customarily used in writing the Slutzky equation:

Total effect = substitution effect + income effect

$$\partial X_2/\partial p_2 \quad = \quad \partial X_2/\partial p|_U \quad + \quad - X_2\partial X_2/\partial M \tag{9}$$

For a normal good, $\partial X/\partial M > 0$, which means that the income effect of a price increase, $-X\,\partial X/\partial M$, is negative, and so the total effect of the price change is greater in magnitude than the substitution effect; i.e., a constant money income demand curve is steeper than the compensated demand curve. For an inferior good, the compensated demand curve is steeper in magnitude and, in the extreme case of a Giffen good, the constant money income demand curve has a positive slope because the positive income effect overwhelms the substitution effect.

[1] For this utility function $\lambda = 1/\sqrt{p_x p_y}$.